1. Scalar: 3, 40, -4
2. **Vector**
   - definition: an ordered list of scalars
   - categories: row vector, column vector
   - special case: zero vector
   - operations:
     - equal: same size + same corresponding elements
     - in R^n (nth real dimension)
     - addition
     - scalar multiplication
     - **dot product**:
       - $\begin{bmatrix}1\\5\\8\end{bmatrix} \cdot \begin{bmatrix}12\\3\\6\end{bmatrix} = (1x12) + (5x3) + (8x6) = 12 + 15 + 48 = 75$
   - Draw a 2-dim vector in R^2 and its addition - the parallelogram rule

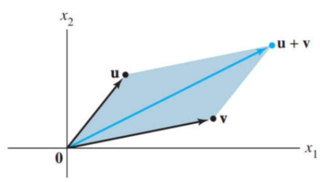
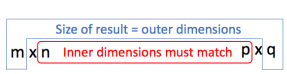


FIGURE 3 The parallelogram rule.

3. **Matrix**
   - definition:
     - a rectangular array of scalars.
     - concatenate row vectors vertically
     - concatenate column vectors horizontally
   - indexing: are given by the row and column number of the element.

$$\begin{bmatrix} 2_{1,1} & 9_{1,2} & 67_{1,3} & 78_{1,4} \\ -5_{2,1} & 64_{2,2} & 21_{2,3} & 12_{2,4} \\ 8_{3,1} & 19_{3,2} & 32_{3,3} & 77_{3,4} \\ 83_{4,1} & 17_{4,2} & 2_{4,3} & -4_{4,4} \\ 28_{5,1} & 11_{5,2} & 54_{5,3} & 19_{5,4} \end{bmatrix}$$

   - main diagonal of the matrix
   - special case: square matrix, **identity matrix**
   - Operations:
     - addition
     - scalar multiplication
     - **transpose:** flip row and column $\mathbf{A}^T$
     - **Matrix multiplication**:
       - constrain: inner dimensions must match



$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{1,m} & \cdots & a_{m,n} \end{bmatrix} \qquad B = \begin{bmatrix} b_{1,1} & \cdots & b_{1,q} \\ \vdots & \ddots & \vdots \\ b_{1,p} & \cdots & b_{p,q} \end{bmatrix}$$

- Matrix power: only possible for square matrices

1. Linear System of Equations
   - Matrix form: Ax=b
   - Augmented matrix: [A b]

$$x_1 + 4x_2 + 12x_3 = 18$$
$$7x_1 + 3x_3 = 21$$
$$6x_1 - 10x_2 + 15x_3 = 45$$

$$\underset{A}{\begin{bmatrix} 1 & 4 & 12 \\ 7 & 0 & 3 \\ 6 & -10 & 15 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 18 \\ 21 \\ 45 \end{bmatrix}}$$

Fig. 1: linear system of equations

2. How to solve the problem?
   - Row Reduction
     - principle: **eliminate** variables in other equations to obtain a simplified **equivalent** system of equations
     - three elementary row operations:
       - **swap** any two rows
       - multiply any row by a non-zero scalar
       - add any scalar multiple of a row to another row
     - **Row Echelon Form**:
       - definition:
         - any rows of all zeros appear at the bottom of the matrix
         - the first non-zero entry (**leading entry**, **pivot entry**) in each row, (reading from the left), occurs to the right of the one in the row above it.
         - all entries in a column **below** a leading entry (pivot entry) are **zeros.**
       - properties:
         - not unique
         - if the augmented matrix is in row echelon form, the system can be solved using back-substitution
     - **Reduced Row Echelon Form**
       - definition:
         - row echelon form
         - every pivot entry is **one**
         - all entries in a column **above** each pivot are **zeros**
       - properties:
         - **unique** for every matrix
         - if the augmented matrix is in reduced row echelon form, then the system can be solved easily by inspection.

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Fig.2: row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Fig.3: reduced row echelon form

3. How many solutions do we have?
   - **Free** Variables & **Basic** Variables
     - variables corresponding to pivot columns are called basic variables
     - otherwise, free variables

$$\left[\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\quad x_1 \quad x_2 \quad x_3 \quad x_4$$

- x_2 and x_4 are basic variables
- x_1 and x_3 are free variables

$$\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Fig.4: inconsistent system

   - inconsistent system: **no** solution
     - a system of linear equations is inconsistent **if and only if** a row echelon form of its augmented matrix has a pivot in the **right most column**
   - consistent system:
     - **unique** solution: **no free** variables
     - **infinite** solution: has free variables

(continue from lecture#2)

4. For linear system of equations, which has infinite solution, how do we **parameterize** all solutions?
    - principle: write all variables in terms of **free** variables
    - solution set = **particular vector** + **linear combination of spanning vectors**
    - there is only one spanning vector for each free variable

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 5 \\ 0 \\ 6 \end{bmatrix}}_{p} + x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + x_3 \underbrace{\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{v_2}$$

5. **linear combination**:
    - definition:
        - A linear combination of vectors is **a weighted sum of** a set of n vectors:[v_1, v_2, v_3, …, v_n], written as: $c_1v_1+c_2v_2+c_3v_3+\ldots+c_nv_n$, where the c values can be any scalar
    - example:

$$c_1\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} + c_3\begin{bmatrix} 3 \\ 4 \\ -7 \end{bmatrix}$$

6. **span**:
    - definition:
        - the span of vectors is **the set of all their linear combinations**.
        - those original vector set are called the **spanning vectors**
    - shape:
        - the set of all points (vectors) we can reach through scaling & addition is the span

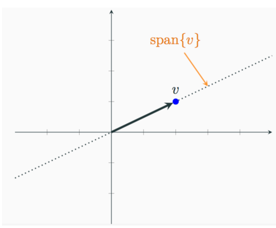


*Fig 1: span of a nonzero vector - **straight line** passing through v and 0*

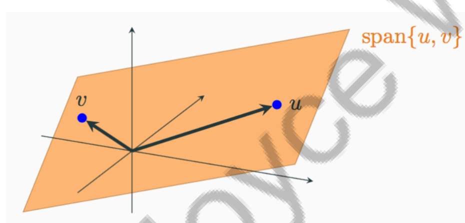


*Fig 2: span of two **independent** vectors - **plane** passing through u, v and 0*

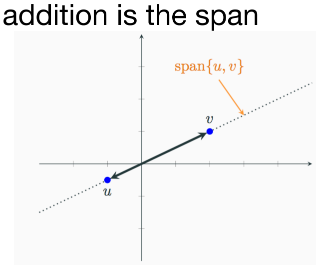
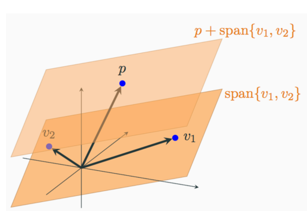
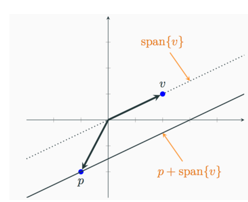


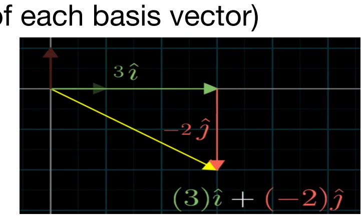
*Fig 3: span of two **colinear** vectors - **straight line** passing through u, v and 0*

7. **Homogeneous Equation** Ax=0
    - **trivial** solution: the zero vector
    - What is the correlation between solution sets of Ax=b (consistent case) and Ax=0?
        - <u>*property#1*</u>: have the same number of solutions
            - reason: #free-variables in rref (reduced row echelon form) are the same
        - <u>*property#2*</u>: if **p** is the particular solution of Ax=b, **v** is solution set of Ax=0 => **w=p+v** is the solution set for Ax=b
        - <u>*property#3*</u>: if **w=p+v** is the solution set of Ax=b, v is the span part=> **v** is the solution set of Ax=0
    - Geometrically: The solution set of Ax=b is a line/plane through **p** parallel to the solution set of Ax=0





8. Geometric understanding
    - vector addition: combining two **movements**, represented by each vector
    - <u>Anytime we describe vector numerically, it depends on an implicit choice of what **basis vectors** we are using.</u>
    - vector is a linear combination of the coordinate system's basis vectors (the value of each entry in the vector corresponds to the coefficient/scalar of each basis vector)



1. Linear Independence
- definition:
    - The vectors in a set $T = \{\overrightarrow{v_1}, \overrightarrow{v_2}, ..., \overrightarrow{v_n}\}$ are said to be *linearly independent* if the equation: $a_1\overrightarrow{v_1} + a_2\overrightarrow{v_2} + ... + a_n\overrightarrow{v_n} = \vec{0}$, can only be satisfied by $a_i = 0$ for $i = 1, ..., n$
    - This implies that **no** vector in the set can be represented as a linear combination of the remaining vectors in the set.
- special case: v_1 and v_2 are called **colinear** $\Leftrightarrow$ one of the vectors is a scalar multiple of the other.
- **row reduction** is the easiest way to tell which vectors in a set are linearly dependent or independent

2. Revisit Ax=b
- if the vector b is inside the span of the columns of A (the matrix equation has a solution), then b can be written as <u>*a linear combination of the columns of A*</u>.

$$\overset{A}{\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} \overset{=}{=} \overset{b}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} \qquad x_1 \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ a_{3,1} \end{bmatrix} + x_2 \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ a_{3,2} \end{bmatrix} + x_3 \begin{bmatrix} a_{1,3} \\ a_{2,3} \\ a_{3,3} \end{bmatrix} = b$$

3. The **essence** of matrix-vector multiplication (Ax=b)
- Applying a certain linear transformation to the vector
    - $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$
    - this linear transformation is completely **determined** by where it takes the **basis vectors** of the space.
    - [(a,c) and (b,d)] are the two basis vectors [(1,0) and (0,1)] moved to after the transformation
- **Identity** transformation:
    - the transformation does nothing, the i-hat and j-hat remains unmoved.

4. Linear transformation
- Numerically: formal linearity properties:
    - T(**u**+**v**) = T(**u**) + T(**v**)
    - cT(**u**) = T(c**u**)
- Geometrically:
    - origin remain fixed
    - <u>lines remain lines without getting curved</u> $\Rightarrow$ <u>keeping grid lines parallel and equally space.</u>
- examples:
    - scaling, rotation, reflection, shear (scale the vector differently in different dimensions), projection
- what is invertible transformation?
    - <u>for **every** $u \in R^m$ there is **one and only one** $v \in R^n$ such that $u = T(v)$</u>
    - Corresponding inverse transformation takes u to v, $T^{-1} : R^m \to R^n$

5. Matrix Inverse
- <u>A^{-1} is the **unique** transformation with the property that if you first apply A, then followed it with the transformation A^{-1}. You end up back where you started.</u>
- $AA^{-1} = A^{-1}A = I$

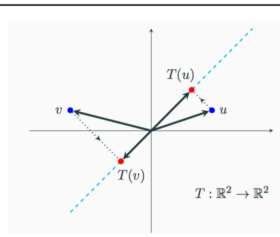


Fig1: Projection. Not invertible

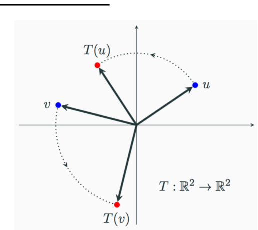
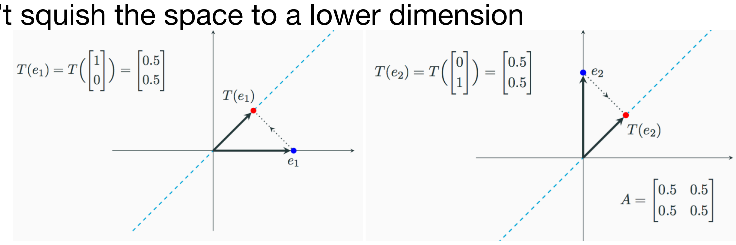


Fig2: Rotation. Invertible

1. Invertible Transformation: onto & one-to-one
   - definition: for **every** $u \in R^m$ there is **one and only one** $v \in R^n$ such that $u = T(v)$
   - onto ⇔ every:
     - every vector b in the transformed space has one/multiple correspondence in the original space
     - Ax=b has solution for every b ⇔ Ax=b is consistent for every b
     - A has a pivot i**n every row**
   - one-to-one ⇔ one and only one: if
     - if b has correspondent mapping in the original space, then that mapping is unique.
     - all variables in the system are basic variables, no free variables
     - A has a pivot **in each column**
   - invertible/isomorphism
     - one-to-one & onto
2. Determinant
   - definition geometrically:
     - <u>tells how much a transformation **scales** area</u>
     - <u>the **factor** by which a linear transformation changes any area</u>
     - the **sign** represents the orientation of the space. If determinant is **negative**:
       - space gets flipped over
       - the orientation of the space has been inverted
       - in 2D case, <u>j-hat is to the right of i-hat</u>
   - compute:
     - 2x2 case $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$
   - properties:
     - if A and B are $n \times n$ matrices, then $\det(AB) = (\det A)(\det B)$
3. Revisit Ax=b: $\det(A)$ & A^{-1}
   - if $\det(A) = 0$:
     - <u>the transformation A **squishes** space into a smaller dimension</u> (think about projection: plane->line, *figure below*)
     - A^{-1} doesn't exist!
       - you cannot un-squish a line to turn it into a plane (not one-to-one)
     - solution might exist
       - if we are lucky enough, that the vector b lives on that transformed line/plane
   - if $\det(A) \neq 0$:
     - <u>space doesn't get squished into a **lower** dimension</u>
     - can always find the **unique** vector x lands on b after the transformation A
     - A^{-1} exist! (one-to-one, onto)
     - solution exist: $x = A^{-1}b$, where $AA^{-1} = I$
   - Note: A^{-1} exist $\Rightarrow Ax = b$ have unique solution; A is not invertible $\neq Ax = b$ has no solution
4. If A is invertible:
   - A is a **square matrix.**
   - A has a pivot in every row ⇔ A has a pivot in every column
   - the columns of A are linearly independent
   - each of the column in A cannot be written as a linear combination of the other columns
   - the transformation A represents doesn't squish the space to a lower dimension
   - <u>determinate of A is not ZERO</u>

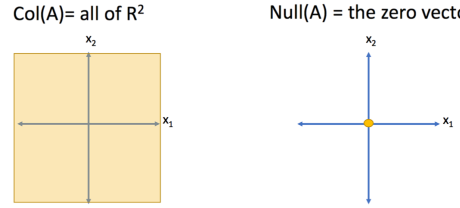


1. subspace, column space, null space and rank.

- subspace: any region within a space R^n that is expressible as a span
    - contains the zero vector
    - is the span of the collection of all vectors
- column space:
    - the **span** of the columns of your matrix, where columns are the basis vectors of the space
- **rank**:
    - the #dimensions in the output of a transformation / in the column space
    - **full rank**: the column space equals #columns, is as high as it can be.
        - columns in standard matrix A are linearly independent
        - A has a pivot in each column
        - for a square matrix A, A is full rank $\Leftrightarrow$ A is invertible (geometrically, the space doesn't get squished after the transformation) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -2 \\ 0 & 0 & -4 \end{bmatrix}$
- null space:
    - the set of vectors that lands on the origin, after the transformation
    - Ax=0, null space gives us all possible solutions to the equation
- nullity:
    - the dimension of the null space of a matrix
- if matrix A has n columns: $rank(A)+nullity(A)=n$

• Example: Graph the columns space of null space of $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

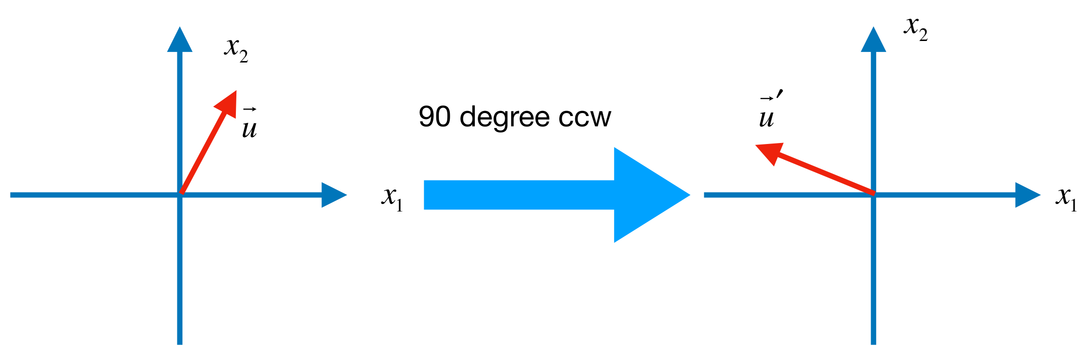
Col(A)= all of R^2 Null(A) = the zero vector



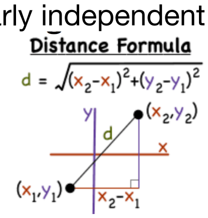
2. How to compute A-inverse?

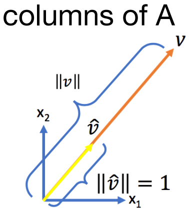
- $[A \mid I] \xrightarrow{rref} [I \mid A^{-1}]$

3. revisit Ax=b, the choice of basis

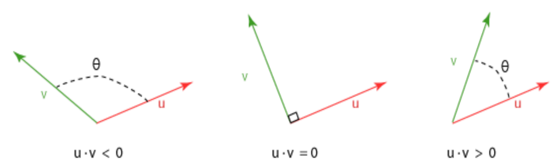
- Ax=b $\Rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$
- x is the coefficients:
    - the linear combination / coordinate of the original vector $\vec{u}$, based on original basis $\hat{i}, \hat{j}$
    - the linear combination / coordinate of the transformed vector $\hat{u}'$, based on transformed basis $\hat{i}', \hat{j}'$
    - the original vector $\hat{u}$ and transformed vector $\hat{u}'$ share the **same** linear combination / coordinate out of two different basis. (as a result of the property: the grid lines remain parallel and evenly spaced)
- b is the linear combination / coordinate of the transformed vector $\hat{u}'$, based on **original basis** $\hat{i}, \hat{j}$
- 

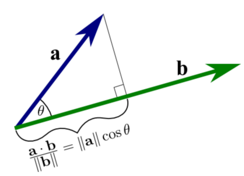
1. Eigenvector & Eigenvalue
   - **Eigenvector** is a special vector that after linear transformation A, it turns into scalar multiples of itself.
     - they are the ones that stay on their own span during a transformation.
   - **Eigenvalue** is the factor by which it is stretched or squashed during the transformation.
   - $Ax = \lambda x \Leftrightarrow (A - \lambda I)x = 0$, x is a **nonzero vector (eigenvector)** and λ is a scalar (eigenvalue)
     - reason: identity matrix multiple any matrix won't change that matrix
   - **Characteristic equation**:
     - $\det(A - \lambda I) = 0$
   - How to compute eigenvalue?
     - homogeneous system $(A - \lambda I)x = 0$ has **nontrivial/nonzero** solution for particular λ
     - matrix $A - \lambda I$ is not invertible $\Leftrightarrow \det(A - \lambda I) = 0$
     - get λ
   - How to compute eigenvector with respect to specific eigenvalue λ?
     - eigenvector is the solution set of the homogeneous system $(A - \lambda I)x = 0$
     - rref of $A - \lambda I$; write all variables in terms of free variables; parametric form.
2. Basis
   - a basis for a subspace is a linearly independent set whose span is the subspace
   - **standard** basis: identity matrix
   - a basis for the column space A is formed from the linearly independent columns of A
3. magnitude of a vector:
   - the vector's length / the distance from tail to head
   - $\|u\| = \sqrt{u \bullet u}$



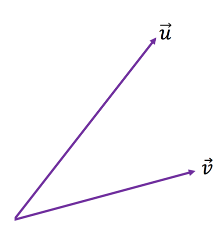


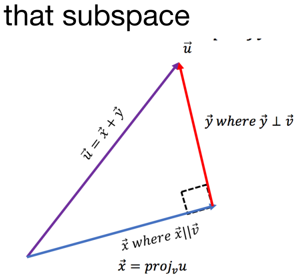
4. unit vector
   - vector whose magnitude is one
   - the unit vector in the same direction of u: $\hat{u} = \frac{u}{\|u\|}$
5. dot product geometrically
   - $a \bullet b = \|a\|\|b\|\cos\theta$
   - if a and b are perpendicular, $a \bullet b = 0$
   - if a and b are **opposing** directions, $a \bullet b \prec 0$
   - if a and b are pointing in the same direction, $a \bullet b \succ 0$



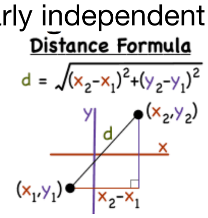


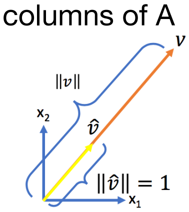
6. orthogonal basis
   - basis vectors are orthogonal to each other
   - the dot product of any two vectors in the set is zero
7. orthonormal basis
   - orthogonal + normalize
   - all vectors in the set are **unit** that are orthogonal to each other
8. **orthogonal matrix** A: A is square matrix with **orthonormal** columns. $A^{-1} = A^T$ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
9. Graham-Schmidt Procedure
   - a method to find an orthogonal basis for a subspace given any basis for that subspace
   - given basis $\vec{u}, \vec{v}$, the orthogonal basis $\vec{x}, \vec{y}$
   - $\vec{x} = proj_v u = \frac{\vec{v}}{\|v\|} \cdot \|u\| \cos\theta = \frac{\vec{u}\vec{v}}{\vec{v}\vec{v}} \cdot \vec{v}$; $\vec{y} = \vec{u} - \vec{x}$



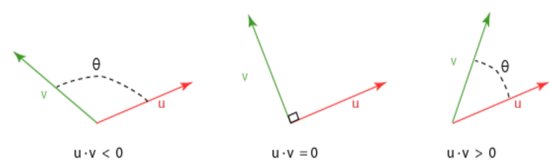


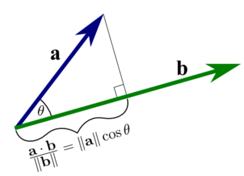
1. Eigenvector & Eigenvalue
   - **Eigenvector** is a special vector that after linear transformation A, it turns into scalar multiples of itself.
     - they are the ones that stay on their own span during a transformation.
   - **Eigenvalue** is the factor by which it is stretched or squashed during the transformation.
   - $Ax = \lambda x \Leftrightarrow (A - \lambda I)x = 0$, x is a **nonzero vector (eigenvector)** and λ is a scalar (eigenvalue)
     - reason: identity matrix multiple any matrix won't change that matrix
   - **Characteristic equation**:
     - $\det(A - \lambda I) = 0$
   - How to compute eigenvalue?
     - homogeneous system $(A - \lambda I)x = 0$ has **nontrivial/nonzero** solution for particular λ
     - matrix $A - \lambda x$ is not invertible $\Leftrightarrow \det(A - \lambda I) = 0$
     - get λ
   - How to compute eigenvector with respect to specific eigenvalue λ?
     - eigenvector is the solution set of the homogeneous system $(A - \lambda I)x = 0$
     - rref of $A - \lambda I$; write all variables in terms of free variables; parametric form.
2. Basis
   - a basis for a subspace is a linearly independent set whose span is the subspace
   - **standard** basis: identity matrix
   - a basis for the column space A is formed from the linearly independent columns of A
3. magnitude of a vector:
   - the vector's length / the distance from tail to head
   - $\|u\| = \sqrt{u \bullet u}$



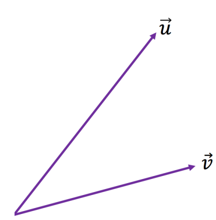


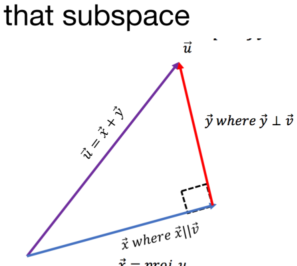
4. unit vector
   - vector whose magnitude is one
   - the unit vector in the same direction of u: $\hat{u} = \dfrac{u}{\|u\|}$
5. dot product geometrically
   - $a \bullet b = \|a\|\|b\|\cos\theta$
   - if a and b are perpendicular, $a \bullet b = 0$
   - if a and b are **opposing** directions, $a \bullet b \prec 0$
   - if a and b are pointing in the same direction, $a \bullet b \succ 0$



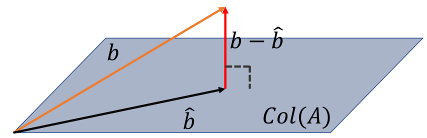


6. orthogonal basis
   - basis vectors are orthogonal to each other
   - the dot product of any two vectors in the set is zero
7. orthonormal basis
   - orthogonal + normalize
   - all vectors in the set are **unit** that are orthogonal to each other
8. **orthogonal matrix** A: A is square matrix with **orthonormal** columns. $A^{-1} = A^T$ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
9. Graham-Schmidt Procedure
   - a method to find an orthogonal basis for a subspace given any basis for that subspace
   - given basis $\vec{u}, \vec{v}$, the orthogonal basis $\vec{x}, \vec{y}$
   - $\vec{x} = proj_v u = \dfrac{\vec{v}}{\|v\|} \cdot \|u\| \cos\theta = \dfrac{\vec{u}\vec{v}}{\vec{v}\vec{v}} \cdot \vec{v}$; $\vec{y} = \vec{u} - \vec{x}$



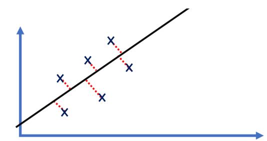


1. **Least Square Approximation**
   - the approximate solution for inconsistent system Ax=b
   - **principle**: b is not in the column space of A, therefore the system has no solution for x.
     But it has approximate solution $\hat{x}$, where the corresponding $\hat{b}$ is closer to b than any other vector in col(A).
   - **objective**: minimize the distance between **real** b (not in col(A)) and **approximate** $\hat{b}$ (in col(A))
     - $\min\|b-\hat{b}\|^2$ where $b-\hat{b}$ is the vector going from col(A) to b



   - **solution**:
     - $\hat{b} = \mathrm{Proj}_{col(A)} b \Leftrightarrow b-\hat{b}$ is orthogonal to col(A)
     - $A^T(b-\hat{b}) = 0 \Leftrightarrow A^T b = A^T \hat{b} \Leftrightarrow$ $\underline{A^T A\hat{x} = A^T b}$ <u>(matrix-vector multiplication)</u>
     - augment matrix $[A^T A \mid A^T \hat{b}]$, do row reduction, get rref and the solution set for $\hat{x}$
     - $\hat{x}$ is the approximate solution for Ax=b

2. **Least Square Regression Line**



   - problem:
     - given a set of points: $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle \ldots, \langle x_n, y_n\rangle$
     - find <u>a regression line</u> (a best-fit line) that attempts to fit a set of points to a straight line. Format the regression line $y = C_1 x + C_0$
   - **objective**: minimize the **mean squared error** between every data point and the line.
     - equals: minimize the average **distance** between a data point and its orthogonal projection on the line
   - matrix format: Ax=b. no solution.
   - find approximate solution using Least Square Approximation (above)

$$\underset{A}{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}} \begin{bmatrix} c_1 \\ c_0 \end{bmatrix} = \underset{b}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}$$

3. **Lease Square Regression Curve**
   - **order** corresponds to the **highest exponent** of x
   - 1st order curve (line): $y = C_1 x + C_0$
   - 2nd order curve (parabola): $y = C_2 x^2 + C_1 x + C_0$
   - 3rd order curve: $y = C_3 x^3 + C_2 x^2 + C_1 x + C_0$
   - nth order curve: $y = C_n X^n + \ldots + C_3 x^3 + C_2 x^2 + C_1 x + C_0$
   - Theory: <u>a set of n data points can be **fit perfectly** with a curve of order n-1</u>
     - e.g.: three points can be fit perfectly by a parabola.

4. Cross product
   - definition: $v \times w = p$, cross product p is a vector
   - magnitude:
     - the area of the parallelogram that vector v, w enclose
     - $\det([v, w])$
   - direction:
     - p is perpendicular to v and w
     - obeys <u>the right hand rule</u> (the forefinger points to v)